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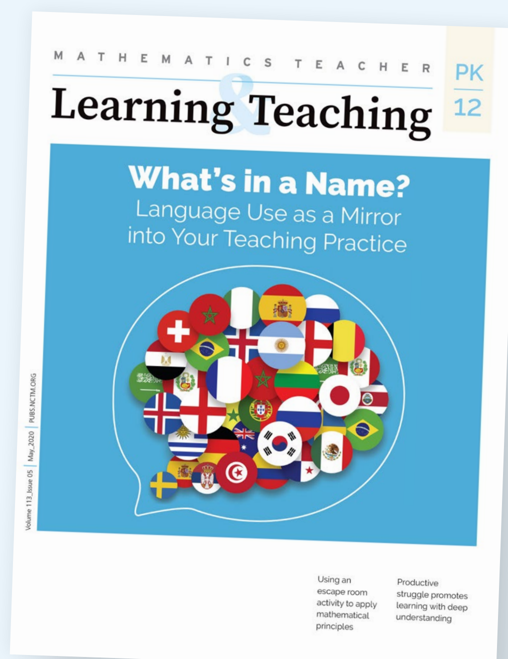

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Understanding Similarity through Dilations of Nonstandard Shapes

This set of tasks progressively engages students in geometric proportional reasoning.

Marina Basu, Karen Koellner, Jennifer K. Jacobs, and Nanette Seago

Using transformations to teach similarity and congruency is a relatively recent requirement within core mathematics standards, and the concept of similarity is important in building mathematical competency across a range of topics, including scale factor, proportional reasoning, and linear functions, among others. Thus, strategies and activities that address conceptual understanding of similarity through geometric reasoning, in addition to or beyond numeric proportional reasoning, warrant close pedagogical attention.

Dilations have been identified as essential in understanding similarity through a dynamic approach (NGA Center and CCSSO 2010). A quick glance at middle school mathematics textbooks shows that two of the most common shapes used to teach dilations are the triangle and the quadrilateral—shapes that students encounter from an early age. It would then make pedagogical sense if a familiar geometric figure was used to scaffold the learning of new or unfamiliar mathematics content. However, what would be the implications if dilations were introduced through familiar yet nonstandard figures instead?

In this article, we discuss a set of tasks that introduce dilations using nonstandard figures, incorporating snippets of classroom conversations as students try to make sense of the specified tasks, to show how they progress in their understanding. We provide a theoretical foundation for the tasks and discuss the pedagogical implications for developing geometric proportional reasoning in students, where visual and spatial senses are invoked. Cox (2013) identifies similarity as the important connector between numeric and geometric reasoning. However, a static definition-led conventional approach to teaching similarity has often relied exclusively on numeric reasoning, leading to procedural rather than conceptual understanding (Seago, Jacobs, and Driscoll 2010).

The evidence of practice included here comes from a sixth-grade class in a middle school in Hawaii. The video footage was collected as part of the Learning and Teaching Geometry (LTG) professional development materials. The task set was created by the Curriculum Research and Development Group, University of Hawaii at Mãnoa (link online). The entire three-task set illustrates a learning trajectory that builds foundational understanding of similarity and congruence (see Seago et al. 2017 for details of the program). The particular tasks that we will be discussing in this article focus on building students' understanding of dilations, which is a process used to create similar figures by enlarging or reducing them.

The three-task set begins with the Heart Stickers, a task that highlights the need for more precise language and definitions about similar figures. The second task, Fruit Punch Spill, highlights the notion that multiple strategies exist for dilating a figure, the preservation of angles, and the scale factor. The third task, Sheri's Method, highlights the fact that for all points in a dilation, the following is invariant: the ratio between the distance from the center of dilation to a point in the original figure and the distance from the center of dilation to the corresponding point in the new figure. Together, these tasks focus on examining preservation of angles and proportional lengths through dilation, including similar irregular figures. We discuss each of the tasks in the sections below.

THE FIRST TASK

The Heart Stickers task includes several different drawings of hearts (see figure 1). This task is intended to highlight issues about proportionality, angles, and using nonstandard shapes to introduce similarity. The task was developed for sixth-grade students, with the expectation that they would draw on their intuitive sense of proportion while recognizing the need for new vocabulary and content knowledge to satisfactorily address the question (Slovin 2000).

The Heart Stickers task elicits children's intuitive understandings (Hollebrands 2004) as they are asked to explain why some of the stickers "don't look right." The notion of "looking right" taps into students' mental images and connects to visualization, which is the first level of geometric thinking (van Hiele 1999). At the same time, "looking right" does not always correspond

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to an accurate conceptual understanding of geometric figures. Students who recognize only a certain orientation of a shape and then categorize different orientations of that same shape as distinct or different fall into van Hiele's level 0 or 1. In this particular context, by including various versions of the same shape (hearts that are similar but not congruent, as well as hearts that are not similar), the question deliberately prompts students to connect to their visual sense of figures and pushes them to notice and try to articulate differences.

Before continuing the discussion, it might be useful to pause and reflect on the following questions as a reader/middle school teacher:

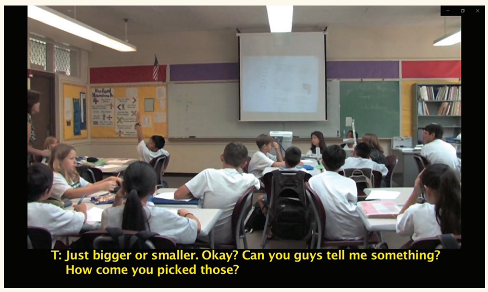
1. Which hearts "don't look right" to you? Why not?
2. What language would you expect sixth graders to use when answering this question?

As teachers, we would likely refer to the concepts of dilation and similarity to explain why certain hearts "don't look right." However, for sixth graders who are not yet aware of these concepts, the question prompts much discussion and can lead to a productive struggle in the classroom. As conceptualized by Warshauer (2015), students are engaged in productive struggle when a task is such that it maintains cognitive demand throughout and teachers address the struggle in a way that supports student thinking. Watching video 1 (link online) might be useful here to see how children try to explain their observations.

When students encounter the first task, intuitively identifying the figures that "look right" is easy for them; however, heart-shape properties are more challenging to describe compared with rectangles or triangles. This task helps students confront the limitations of the terminology they have readily available to explain the differences between the hearts. In the videotaped class, students used informal and largely imprecise language in their efforts to contrast the hearts (see video 1 [link online]). Initially, students used the word *stretched* to refer to some of the hearts that are elongated either horizontally or vertically. But then the teacher drew their attention to the hearts that were not stretched and how to describe them.

By pushing students to consider what features of the hearts helped them determine if they were the same shape, the teacher's goal was to get them to articulate what they meant by "same." As she pointed out, different rectangles are all the same geometric figure, and yet we might not want to say they have the same shape, to which another student responded: "The same—the same, like, it

Video 1 Students Working on the Heart Stickers Task



Watch the full video online.

Fig. 1

The first task, the Heart Stickers, highlights the need for more precise language and definitions about similar figures.

looks like a heart, and it's, like, the same. I can't think of the word!" The students reached an intentional impasse when their existing mathematics vocabulary was no longer sufficient to explain a solution to the task. The teacher informed the students that they would continue to grapple with these ideas as they worked on additional tasks that, by design, take advantage of the students' need for more precise language and definitions.

In effect, the first task sparks the need for students to learn about dilation, which they will through the next two tasks that progressively build toward formalizing the relevant geometrical concepts. Before we describe the other tasks in the sequence, let's first discuss the significance of using the Heart Stickers task to launch the unit.

Why the Heart Stickers Task?

As teachers, if we are encountering the Heart Stickers task for the first time, we might have several questions about the task, including the following:

- What is the importance of using a nonstandard shape like the heart?
- What does this particular task add to the pedagogy on dilations and transformations in general?

The Heart Stickers task is a mathematically rich task. Rich tasks are mathematically meaningful "low threshold/high ceiling" tasks (McLure 2011), and thus enable all learners to engage with the subject productively. The task has many approaches, and students can engage in meaningful math talk to make sense of the task. Importantly, Cox and Edwards (2012) explain that scaling traditional figures like triangles and rectangles neither pushes students to go beyond the rote application of procedures nor capitalizes on their visual skills. On the other hand, using complex shapes like hearts (or crowns, etc.) "helps students develop more robust strategies and investigate more global rules for scaling . . . and pushes them to see the relationships between scaled images beyond numeric proportion" (p. 234).

Scaling based on numeric proportion can become primarily a manipulation of numbers, leading students to understand dilations in a static rather than dynamic manner. To fully understand the geometry of transformations requires engaging with the concepts dynamically, visualizing the movements involved in a dilation (or rotation, reflection, and translation), which can then become formalized through the relevant definitions and axioms (Seago, Jacobs, and Driscoll 2010). Whereas a numeric *proportional thinker* understands the "mathematical characteristics of proportional situations," a *geometric proportional thinker* understands similarity, recognizes whether a shape is distorted or not, and attends to the principles of scaling visually (Cox 2013, p. 9). Moreover, geometric proportional thinkers are not confused by shapes with different orientations, because they understand that similar figures can be translated, rotated, or reflected.

Textbook materials on similarity and congruence often start with definitions of the terms, which is problematic on two counts. First, as Freudenthal (1971) asserted, starting from definitions positions geometry as a reified body of knowledge and removes the experience of mathematics as an activity in which students actively engage in constructing their own knowledge. Thus, the Heart Stickers task does not define dilations or similarity; rather, it brings students to a point where they begin to feel the need for such definitions in order to answer genuine questions that they have. Second, as Wu (2017) argued, textbook definitions are often imprecise, resulting in students' lack of coherence and reasoning abilities while learning the topic (p. 72). Instead, developing conceptual understanding is important, along with a precise vocabulary. Building on students' prior knowledge and formalizing their learning through precise definitions become the subsequent steps in the series of well-chosen tasks described here that follow the first task.

THE SECOND TASK

Whereas the Heart Stickers task highlights the need for vocabulary that would enable students to discuss properties of similar figures and dilations, Fruit Punch Spill (see figure 2) actively engages students in scaling geometric shapes, again through nonstandard ones. However, in contrast to the heart, the arrow shape is a polygon with easily measured side lengths, which is now an appropriate next step for these students. The second task is intended to push on students' deepening understanding of proportion by having them invent a method to enlarge a given figure, without yet knowing the procedural rules or definitions for scaling.

Moving from the first task, in which students are encouraged to notice what is the same and what is different about the heart shapes and find the appropriate language to express their ideas, the second task explicitly refers to "enlargement." Yet, as the task indicates, students do not have definitions or a known set of procedures that will help them find the answer—they need to figure out the principles involved in enlarging an image or a shape.

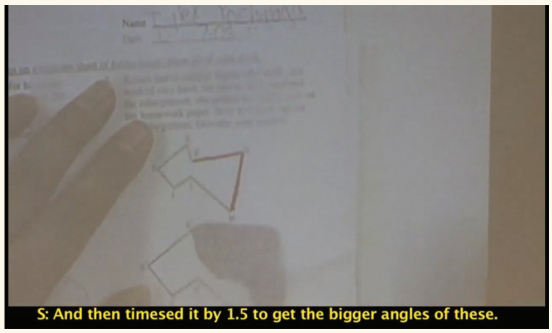
Video 2 (link online) shows how this group of students made sense of the Fruit Punch Spill task and how the teacher pushed them to clarify their thinking and use mathematically precise vocabulary. Of particular significance is the articulation of the idea that “corresponding angles had to be congruent,” and the move toward a conceptual understanding of the scale factor. Discussion of the scale factor was generated from a student’s question about the units of measurement:

Taylor: What unit are you measuring it in?
Teacher: So, let’s see. They took this figure. They took this length. So, what is their unit?
Student: 1.5
Teacher: What is their unit?
Student: Lines
Teacher: What are we counting as one? This length, right? Which is *KE*. So, is it the same? Is our unit we’re using the same?
Students: No.
Teacher: Or does it change?
Students: It changes.
Teacher: It changes, OK? So, we’re using the length of the original sides, right? And then we’re making that as our unit to measure the corresponding side.

As video 2 (link online) shows, students used tracing paper to trace specific sides of the original figure and then applied a scale factor of 1.5 to find the lengths of the enlarged figure. This teacher frequently provided her students with tracing paper as a measurement tool, and here she used it as an aid in developing their spatial and geometric reasoning.

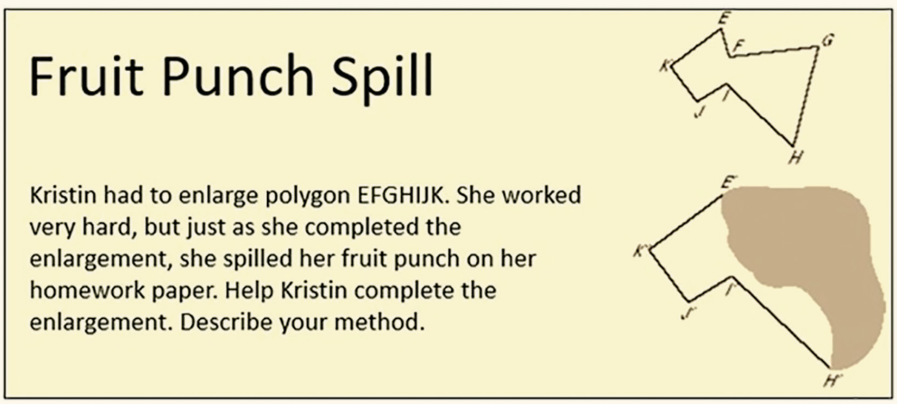
By tracing, students visually identified the unit or quantity and then experienced motion and the dynamic nature of transformations first-hand, as they iterated the unit and engaged with the task using their visual and tactile senses (Cox and Edwards 2012). Perhaps due to their familiarity with tracing paper as a mathematical tool, the videotaped students readily folded the unit length in half to determine that the scale factor had to be 1.5 (i.e., one unit length plus one-half unit length). This action helped students directly grapple with the

Video 2 Using Tracing Paper to Find the Corresponding Angle in Fruit Punch Spill



Watch the full video online.

Fig. 2



The second task actively engages students in scaling geometric shapes through nonstandard ones and highlights the notion that multiple strategies exist for dilating a figure, the preservation of angles, and the scale factor.

scale factor as a multiplicative relationship rather than focusing on the more additive notion of finding the measurements through the conventional use of a ruler. By using multiplicative instead of additive reasoning, students were building their knowledge of proportionality and varying quantities to characterize the two arrow shapes as having a proportional relationship. The teacher helped them arrive at the important insight that the unit (which in this case is the length of each side of the original figure) can change; and unlike the scale factor, the unit (or side length) is not a fixed quantity. By this point, students had grappled with the scale factor (even if the terminology was not explicitly introduced yet) and were ready to engage with tasks that would help them gain a more sophisticated understanding of dilations. They were also in a position to explain the previous Heart Stickers task through the vocabulary of enlargement with or without distortions.

Thus, this task sequence supports progressive development in the understanding of similarity through geometric proportional reasoning rather than numerical reasoning. The nature of the second task also invokes the explicit engagement of a visual strategy in conceptualizing proportional growth (Cox 2013). The third task then builds on students' emerging understandings of enlargement and takes them further along in visually conceptualizing dilations as continuous scaling.

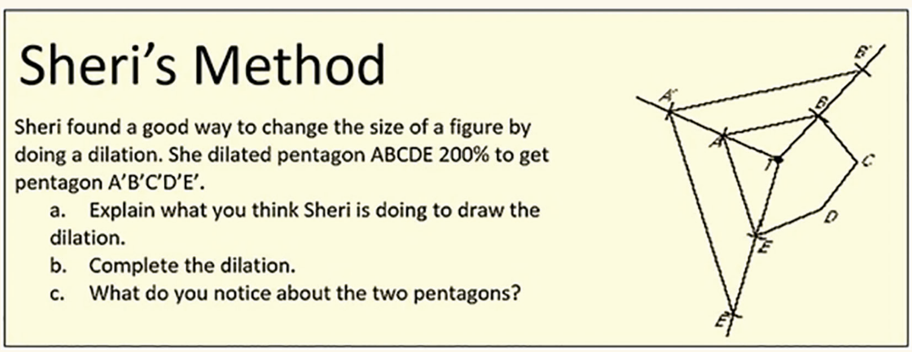
THE THIRD TASK

Sheri's Method (see figure 3) moves students to a more nuanced understanding of dilation and the mathematical properties of dilated figures. This task explicitly uses the term *dilation*, and from the context of the task, students should be able to infer that dilation relates to a change of size. Yet, there is neither a formal definition nor a set of procedures given to help students solve a dilation before they encounter this task. Instead, by providing a partial solution, Sheri's Method encourages active engagement and discovery of one way to dilate a figure. Three lines of dilation are in the given diagram, and completing the task necessitates the drawing of two more lines of dilation (through the remaining two vertices). Students must recognize that to do the dilation, the distance from the center to each vertex of the new pentagon has to be doubled. Marking and joining the vertices creates the dilated pentagon.

Sheri's Method provides students with an implicit introduction to a number of critical concepts, including the center of dilation, lines of dilation, and the necessity of having congruent corresponding angles along the lines of dilation. The teacher emphasized the latter point in her discussion by drawing students' attention to the angles being translated along the lines of dilation, which explains why the angles are the "same" or congruent. See video 3 (link online).

The teacher used the figure, along with gestures and a verbal explanation, to help students understand visually how dilation lines can be drawn from the center of a figure, and that translating a given angle along a dilation line means that the angle will not change (see figure 4). Subsequent discussions between students showed their intuitive recognition that if they failed to preserve the ratios within the shapes, the shapes would no longer be similar.

Fig. 3



By providing a partial solution, the third task encourages students' active engagement and discovery of one way to dilate a figure.

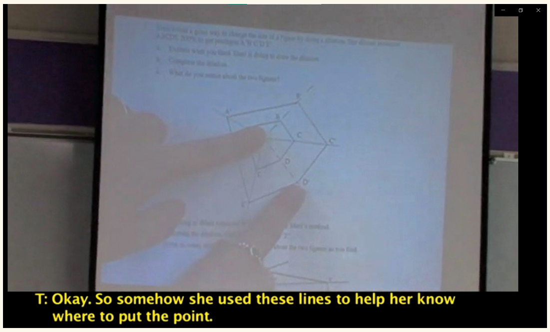
CONCLUSION

The three-task sequence described in this article takes students along a trajectory of learning about geometric similarity through dilations, beginning with an informal task using nontechnical language and a nonstandard shape. The sequence is designed in a manner that allows students to “mathematize,” which “relates directly to the idea of reinvention, a process in which students formalize their informal understandings and intuitions” (Cobb, Zhao, and Visnovska 2008, p. 105). A more traditional approach to this content would likely begin with formal definitions and specified procedures. In contrast, the above sequence of tasks first creates the need for the vocabulary of dilations, and then progressively introduces formal terms and processes, gradually leading students to understand dilations as transformations where angles and proportional lengths are preserved. Thus, the “behaviors of a geometric proportional thinker” (Cox 2013) are developed, which include—

- knowing how to scale images quantitatively and qualitatively and recognizing the continuous nature of the scaling function; and
- being unaffected by the complexity or simplicity of the figure, the relationship of the labeled measurements, and the integral or nonintegral nature of the numbers in the task (Cox 2013, p. 9).

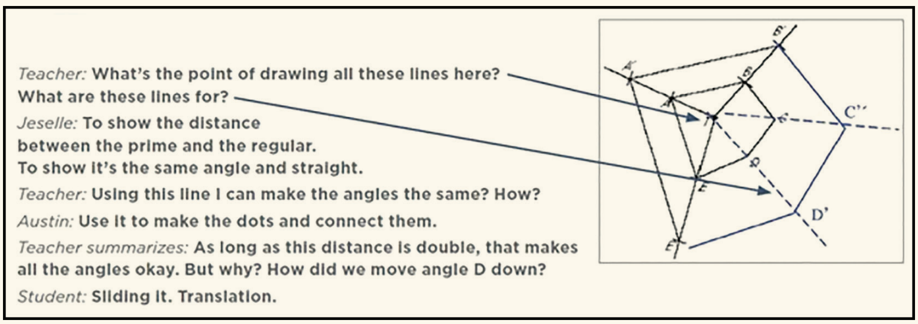
Given this context, it is important for teachers to strengthen their mathematical knowledge for teaching similarity through dilations and have ready access to rich tasks that allow students to become competent in geometric proportional reasoning. The set of tasks discussed here can provide teachers with some of the necessary resources for this endeavor.

Video 3 Discussing the Lines of Dilation in Sheri’s Method



Watch the full video online.

Fig. 4



The class discussed translation and corresponding angles in Sheri’s Method.

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